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Two ways of organizing scalar product in the boundary state method

D. A. Ivanychev
Lipetsk State Technical University (Lipetsk, Russian Federation)

Introduction. The influence of two ways of organizing scalar product on the convergence rate of the solution in the energy method of boundary states is considered. The method is based on the spaces of internal and boundary states which are conjugated through isomorphism. Both spaces are orthonormalized using one scalar product or another. The desired state is expanded in the Fourier series according to the elements of the orthonormalized basis; and the coefficients of this linear combination are determined. The two methods differ in the assignment of scalar products and the calculation of the Fourier coefficients.

Materials and Methods. In relation to the method of boundary states, a new theory of organizing a scalar product in the spaces of internal and boundary states is proposed. Computational algorithms are constructed for its practical implementation. In the traditional (first) approach, the internal energy of elastic deformation is used as an orthogonalizer in the space of internal states. Here, the Fourier coefficients are the work of given forces on the basis vectors of displacement of the boundary points. In the studied (second) approach, scalar products are integrals of the cross products of the basis force vectors at the boundary. Accordingly, the Fourier coefficients are calculated as integrals of the product of the given forces at the body boundary by the basic force vectors.

Results. A numerical study of the first primal axisymmetric problem of the elasticity theory for a transversely isotropic cylinder in the absence and presence of mass forces is conducted. In the absence of mass forces, an analysis of the elastic fields obtained for the same number of used basic elements has shown that the second method has the greatest accuracy of the results. Under solving the problem with the presence of mass forces, the second method did not show efficiency in terms of the uniqueness of the solution; however, it is quite suitable for constructing a multitude of elastic fields used to solve more complex problems.

Discussion and Conclusions. The results obtained can be used to solve boundary-value problems of mechanics of not only an anisotropic body, but also an isotropic one. When solving more complex problems, such as contact and mixed ones, the issue of the rate of convergence requires a separate study.

Keywords: boundary state method, scalar product, internal energy, state spaces, the first main task, mass forces.

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Introduction. The boundary-value problems of the elasticity theory in mechanics are sufficiently studied; therefore, in recent years, case studies have been carried out. For example, the first primal axisymmetric problem for a half-strip [1] is considered, the solution of which is constructed in the form of expansions in systems of Fadl-Papkovich functions and has an explicit form. A general method for solving the first primal problem of the elasticity theory for a rectilinear anisotropic body in the case of plane deformation is proposed [2]. Closed systems of boundary value problems similar to Hilbert's problems are used, which makes it possible to achieve a more general method. Plane isotropic problems are solved using the finite element method based on the Castigliano variational principle [3]. This has provid-

ed for obtaining stress fields on grids of a sufficiently low dimension including those for incompressible materials. Contact problems on the implementation of elliptical stamps in a transversely isotropic elastic half-space are solved [4].

The method of boundary states under solving boundary value problems for anisotropic bodies also found its application. For example, the elastic equilibrium of a transversely isotropic cylinder under the action of axisymmetric surface forces is considered [5]. The problems of torsion of extended cylinders of a material with general anisotropy are studied [6]. A mathematical model is shown for obtaining explicit parametric solutions for isotropic and anisotropic bodies [7, 8], where the medium constants are included as parameters in the elastic fields. A technique has been developed for solving problems of the elasticity theory using computer algebra [9]. In solving problems of a stress-strain unbounded elastic medium containing spherical cavities or inclusions, the boundary-state method was used under different conditions [10].

In this paper, we study two approaches to the assignment of a scalar product in the "body" of the boundary state method. In this case, each state is tested using the example of solving the first primal problem of the elasticity theory. In each task, the same number of used elements is held and the error level is estimated.

Materials and Methods. The boundary state method (BSM) [11] is an energy one; it uses the fundamental theory of series to solve the basic problems of mechanics. The concepts of internal and boundary states are used as supporting ones.

The internal state ξ is due to a set of displacement vector u, strain tensor ε, and stress tensor T:

$$\xi = \{u, \varepsilon, T\} . \tag{1}$$

The boundary state is determined by a set of displacement vector of the boundary points u^v and forces p on the body boundary:

$$\gamma = \{u^v, p\} .$$

The totality of such states forms the basis of the spaces of internal $\Xi = \{\xi_1, \xi_2, \xi_3, ..., \xi_k, ...\}$ and boundary $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, ..., \gamma_k, ...\}$ states. Next, orthogonalization of state bases is carried out, where the following expression is used as an orthogonalizer in the basis of boundary states:

$$(\xi_i, \xi_j) = \int_V \varepsilon_i T_j dV ,$$

in the basis of boundary states – the expression:

$$(\gamma_i, \gamma_j) = \int_S u_i^v p_j dS . \tag{2}$$

A single element $\xi_k \in \Xi$ corresponds to each element $\gamma_k \in \Gamma$, besides, this correspondence is one-to-one: $\xi_k \leftrightarrow \gamma_k$. This enables to reduce finding the internal state to the construction of a boundary state isomorphic to it. In the case of the first primal problem, the desired internal and boundary states are the Fourier series:

$$\xi = \sum_{k=1}^{\infty} c_k \xi_k \ ; \ \gamma = \sum_{k=1}^{\infty} c_k \gamma_k \ , \tag{3}$$

here, c_k are the Fourier coefficients:

$$c_k = \int_S p\, u_k dS , \tag{4}$$

where p is the vector of given surface forces; u_k is the displacement vector in the k-th basic element of the basis of internal states. In this case, basic sets are formed on the ground of a general or fundamental solution to the problem.

The first general solutions to the Lame equation of the linear elasticity theory were built back in the 30s of the last century. The Lame equation is the Euler equation of motion (in this case, equilibrium):

$$\nabla T + f = 0 ,$$

where T is stress tensor; ∇ is Hamilton operator acting as a divergence; f is mass forces.

In the Lame equation, the stress tensor T in accordance with Hooke's law is represented through the strain tensor ε . In turn, the strain tensor in accordance with the Cauchy relation is represented through the displacement vector u. In general solutions to the Lame equation, the displacement vector is determined: through the harmonic vector B and the harmonic scalar in the theory of isotropic elasticity, and through the stress function F — in the theory of anisotropic elasticity.

Vector B (function F) can be represented as a series of basic vectors $B_k = B_k(\alpha^i)$ — coordinate functions α^i. As a result, the following basic elements will be associated with each harmonic basis vector B_k (function F_k):

- displacement vector u_k ;
- strain tensor ε_k ;
- stress tensor T_k ;
- mass forces vector f_k (from equilibrium equations);
- surface forces vector (from Cauchy fundamental relationship):

$$p_k = n \cdot T_k \,,$$

where **n** is the external unit normal to the surface of the body.

According to the listed basic elements, the corresponding vectors or tensors are expanded in Fourier series with the same coefficients c_k , which are determined from the conditions of orthogonality of the basic functions. For example, for the first primal problem in the absence of mass forces, when external forces p are given on the entire surface of the body *S* and orthogonalization of the basis vectors

p_k ($\int_S p_i p_j dS = \delta_{ij}$ is Kronecker delta) is implemented, the coefficients c_k are determined from the expression:

$$c_k = \int_S p\, p_k dS \,. \tag{5}$$

This expression follows from the representation:

$$p = \sum_{k=1}^{\infty} c_k p_k \,.$$

Thus, we study the solution generation method using the expression for scalar products in the basis of boundary states:

$$(\gamma_i, \gamma_j) = \int_S p_i p_j dS \tag{6}$$

and the expression for Fourier coefficients (5).

In the case of the second primal problem, there are the dependencies:

$$(\gamma_i, \gamma_j) = \int_S u_i^v u_j^v dS \,;$$

$$c_k = \int_S u\, u_k^v dS \,,$$

where u is the given displacement vector of points of the body boundary; u_k^v is the displacement vector in the *k*-th base element of the boundary states basis.

Research Results. Parameters of the convergence rate of series and the result accuracy are considered using the example of solving the problem of the elastic equilibrium of a transversely isotropic cylinder of dark-gray siltstone [12] in a dimensionless form (Fig. 1).

The boundary conditions are:

$$p = 0,\ S_1 \big|\ z = -2,\ 0 \le r \le 1 \,;$$

$$p = 0,\ S_2 \big|\ z = 2,\ 0 \le r \le 1 \,;$$

$$p_r = 4 - z^2,\ p_z = 0,\ S_3 \big|\ r = 1,\ -2 \le z \le 2 \,.$$

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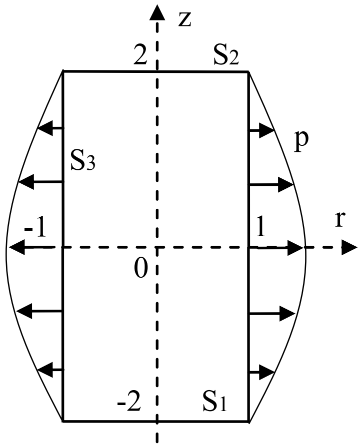

Fig. 1. Boundary conditions for the transtropic cylinder

The methodology for solving the first primal problem in the traditional way using the scalar product (2) is described in detail in [6]. The basis of internal states (1) is constructed as follows:

• using a general solution to the plane deformation problem [13], basis sets of plane auxiliary states are constructed;

• basis sets of spatial axisymmetric states are determined from the transition formulas;

• orthonormalization of the internal states basis is carried out according to the Gram-Schmidt matrix algorithm using the scalar product (2);

• the orthonormal basis of boundary states is reduced from the orthonormal basis of internal states;

• Fourier coefficients (4) are calculated, and series (3) are constructed in an expanded form (index k is placed on top):

$$u_i = \sum_{k=1}^{\infty} c_k u_i^k \ ; \ p_i = \sum_{k=1}^{\infty} c_k p_i^k \ ; \ \sigma_{ij} = \sum_{k=1}^{\infty} c_k \sigma_{ij}^k \ ; \ \varepsilon_{ij} = \sum_{k=1}^{\infty} c_k \varepsilon_{ij}^k \ .$$

We omit the information on the fields of the stress-strain state characteristics obtained with one and the other method of assigning the scalar product, and we give only the main results. In addition, we call the traditional approach used in [11] as the first method for solving the problem, and the approach using the scalar product (6) and Fourier coefficients (5) as the second method. The solution accuracy while keeping the same number of basic elements is higher in the second method. In Fig. 2, a comparison of the obtained boundary conditions (BC) with the specified ones when using the 8 values of the Fourier coefficient is given for each method. The efforts are shown to scale, for example, the true value p_r in the first graph of Fig. 2 is equal to the value on the graph multiplied by the coefficient κ .

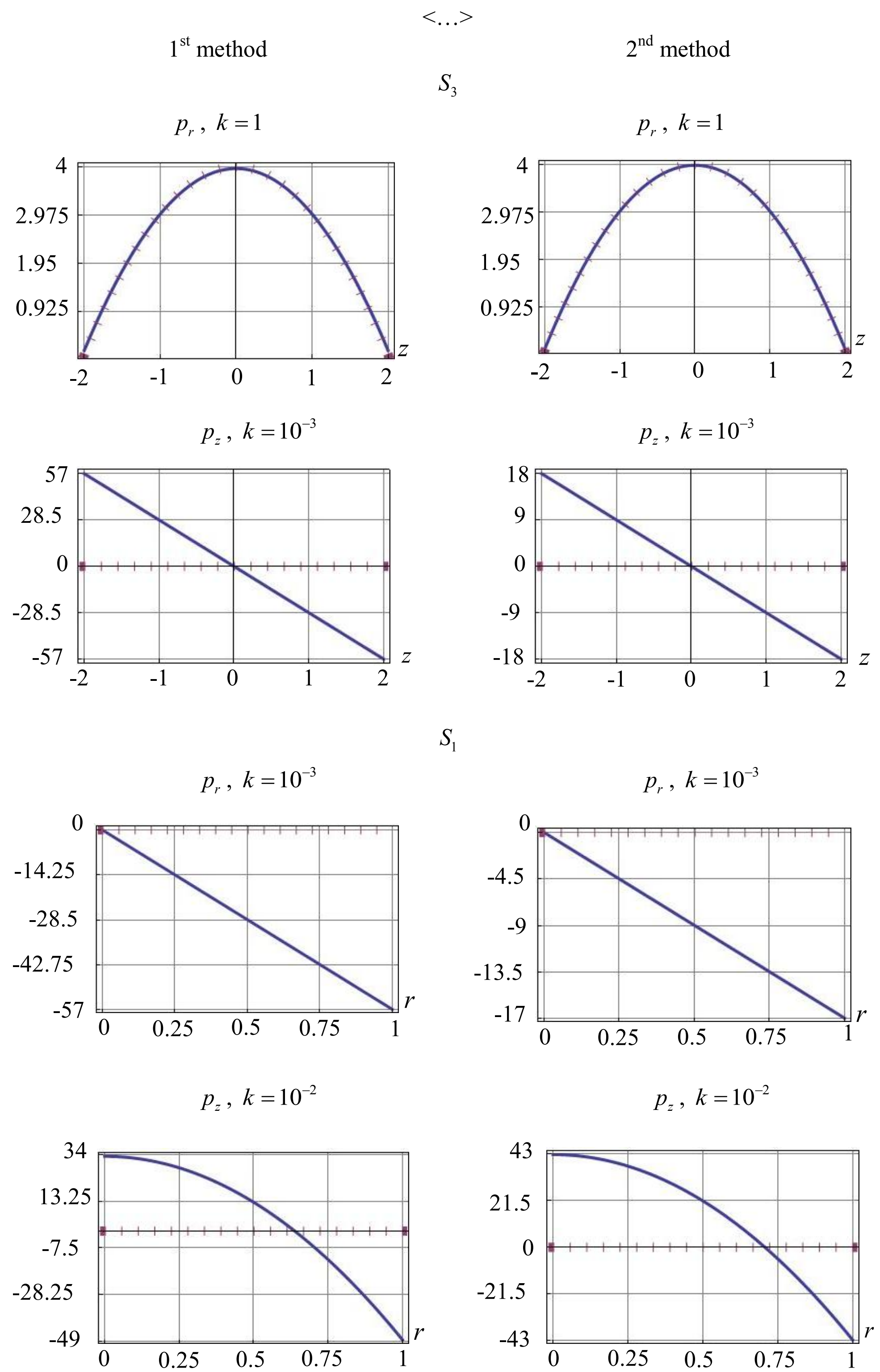

Fig. 2. Verification of BC for cylinder with 8 basis elements

This trend is preserved with an increase in the number of basis elements used. For the 61st element, verification of the boundary conditions is presented in Fig. 3 (comparison of the force p_z on the border section S_3 is given). If we evaluate an error as the maximum deviation of the obtained value from the given value, then in the second method the error is less.

<…>

1st method | 2nd method

S_3

p_z, $k = 10^{-5}$ | p_z, $k = 10^{-5}$

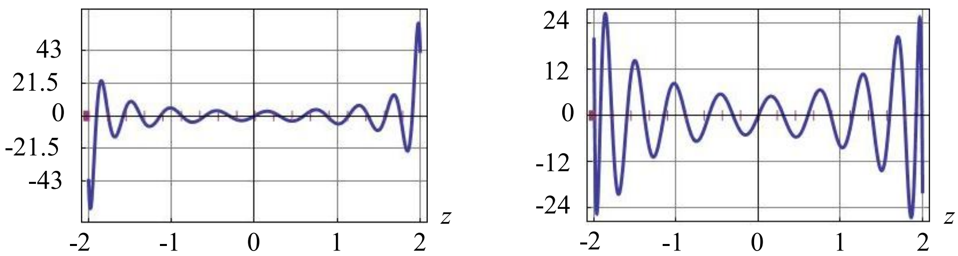

Fig. 3. Verification of BC for cylinder with 61-st basis element

Next, we examine the solution accuracy for a transversely isotropic axisymmetric body of noncanonical shape (Fig. 4). Border conditions are:

$$p = 0,\ S_1 \cup S_2;$$

$$p_r = 0, p_z = 0{,}25,\ S_3 \big|\ z = 1,\ 0 \le r \le 1;$$

$$p_r = 0, p_z = -1,\ S_4 \big|\ z = -1,\ 0 \le r \le 0{,}5.$$

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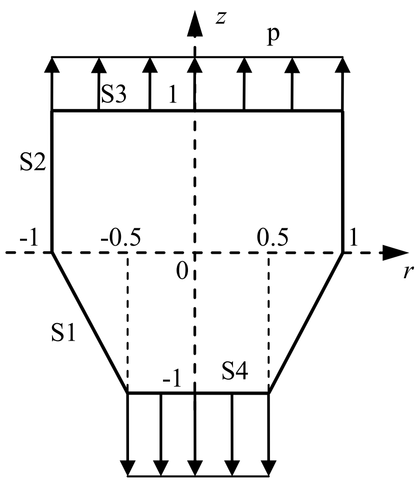

Fig. 4. Boundary conditions for axisymmetric body

15 elements of the basis are kept. Fig. 5 presents a comparison of the boundary conditions for each method (not all boundary sections and components of the force vector are shown).

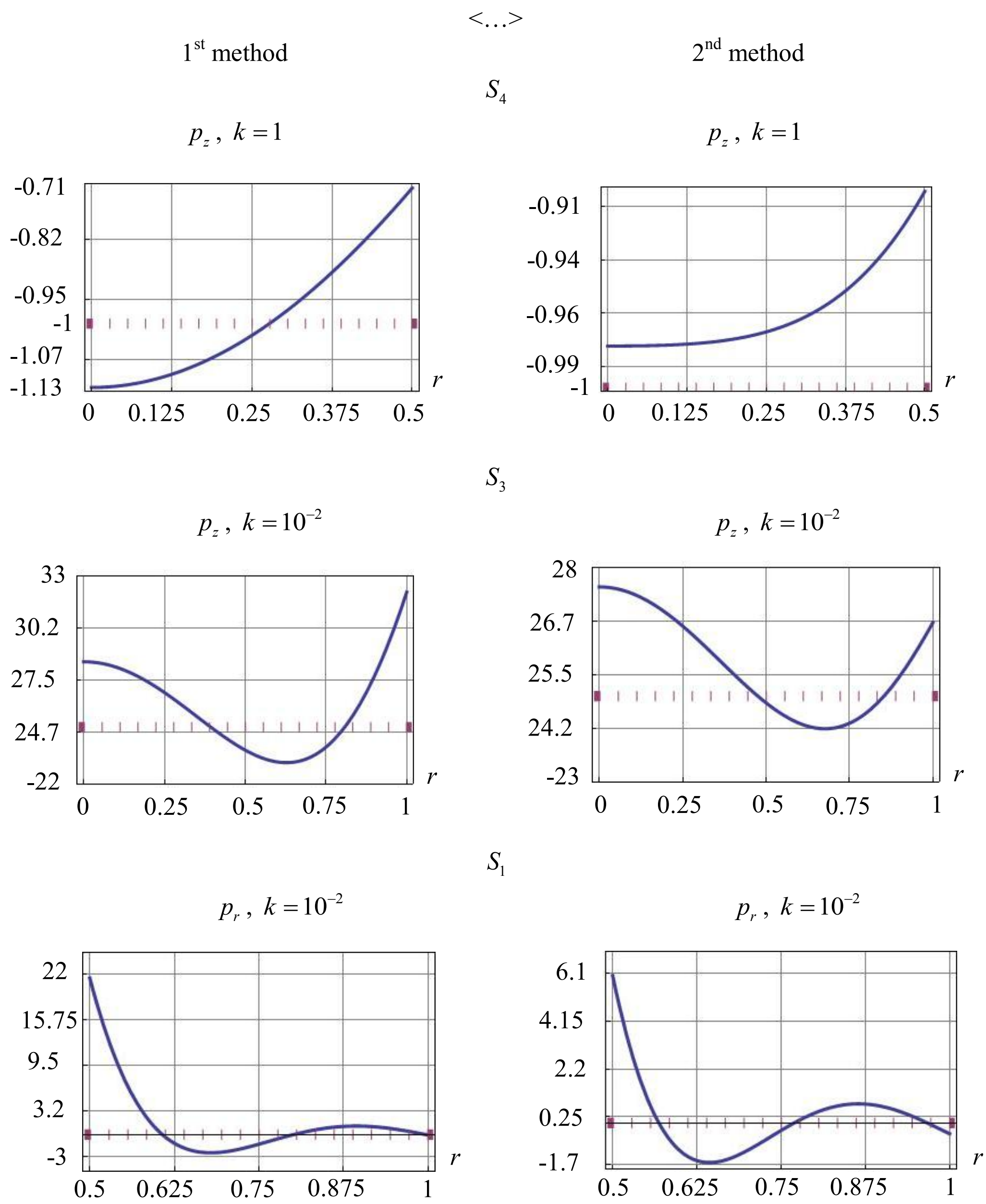

Fig. 5. Verification of BC for axisymmetric body

As can be seen from the graphs, for a body of a more complex shape, a difference in the convergence rate is observed in favor of the second method. Consider tasks involving mass forces. The sequence of the solution generation is as follows:

1. The dependence of the displacement vector of the planar auxiliary state on the coordinates $y^{\alpha} z^{\beta}$ is specified and, on its basis, the displacement vector of the spatial axisymmetric state is determined.
2. For such a vector, the following parameters are determined:
   - strain tensor according to the Cauchy relation;
   - stress tensor from Hooke's law;
   - efforts on the body surface from the fundamental Cauchy relation;
   - mass forces from the equilibrium equation.
3. An exact particular solution to the problem corresponding to the displacement function specified at each point of the body is constructed.
4. Sorting $\alpha + \beta \leq n$ (n = 1, 2, 3 ...), a set of exact particular solutions of the problem of the linear theory of elasticity for the parameters is constructed:

- displacement vectors;
- strain tensors;
- stress tensors;
- vectors of surface forces;
- vectors of mass forces.

5. The bases of internal $\Xi = \{\xi_1, \xi_2, \xi_3, ..., \xi_k, ...\}$ and boundary $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, ..., \gamma_k, ...\}$ states in which the equalities are respected, are generated:

$$\xi_k = \{u_k, \varepsilon_k, T_k\},$$

$$\gamma_k = \{u_k^v, p_k, X_k\}.$$

6. We leave only linearly independent among these solutions and carry out their orthogonalization in accordance with scalar products in the bases of internal and boundary states:

$$(\xi_i, \xi_j) = \int_V \varepsilon_{ij}^{(i)} \sigma_{ij}^{(j)} dV,$$

$$(\gamma_i, \gamma_j) = \int_S p_i^{(i)} u_{vi}^{(j)} dS + \int_V X_i^{(i)} u_i^{(j)} dV$$

(indices *i* and *j*, responsible for the numbers of elements, are placed on top and are enclosed in brackets).

7. As a result, we obtain a basis through which the corresponding vectors or tensors are expanded to series (3) with equal coefficients:

$$c_k = \int_V X u_k dV + \int_S p u_k^v dS,$$

where X is the vector of given mass forces.

We study the possibility of constructing an elastic field in the presence of mass forces using relations (6) and (5). Consider the first primal problem with unbalanced forces for a transversely isotropic cylinder (Fig. 6). Border conditions are:

$$p = 0,\ S_1 \cup S_3;$$

$$p_r = 0, p_z = r^2,\ S_3 \big|\ z = 2,\ 0 \le r \le 1.$$

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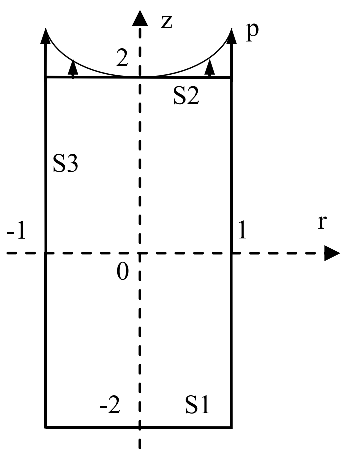

Fig. 6. Boundary conditions for cylinder

The expressions for the orthonormal basis set of the displacement vector components $u = \{u, w\}$ are given in Table 1.

Table 1

Orthonormal basis set of displacement vector components

	u	w
ξ_1	0	$0.1213z$
ξ_2	$0.0788r$	$-0.1126z$
ξ_3	0	$0.0331z^2$
ξ_4	$0.0592rz$	$-0.0268z^2$
ξ_5	$-0.02rz$	$0.0975r^2+0.009z^2$
ξ_6	$0.1535r$	$-0.9163z+0.0691z^3$
ξ_7	$-0.2896r+0.0724rz^2$	$1.2067z-0.1005z^3$
ξ_8	$1.8629r-0.4824rz^2$	$-8.2447z+0.4824r^2z+0.6701z^3$
ξ_9	$0.0908rz$	$-0.0454r^2-0.4347z^2-0.0511z^4$
ξ_{10}	$-0.1757rz+0.0439rz^3$	$-0.0195r^2+0.3661z^2-0.0457z^4$
ξ_{11}	$0.2881rz-0.0804rz^3$	$-0.144r^2-0.7209z^2+0.1206r^2z^2+0.0837z^4$

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To obtain a rigorous solution, it took 11 Fourier coefficients, non-zero values: $c_1=0.2178$; $c_2=-0.1226$; $c_3=0.2377$; $c_4=-0.0732$; $c_5=0.0247$; $c_8=0.1443$; $c_{11}=0.1443$. We give the expressions for displacements and mass forces (4 decimal places are kept):

$$u=0.2592r+0.0367rz-0.0696rz^2-0.0116rz^3;$$

$$w=-0.0183r^2-1.1497z+0.0696r^2z-0.094z^2+;$$

$$+0.0174r^2z^2+0.0967z^3+0.0121z^4;$$

$$R=-0.2814r-0.1407rz;\ Z=1.2012-0.25r^2-3.6038z-0.9z^2.$$

If we construct a basis of internal states through planar auxiliary states formed using monomial $z^\alpha y^\beta$, then the orthonormal basis and the Fourier coefficients change. In this case, the decision will also be rigorous and will be accepted in:

$$u=0.0363r-0.0913rz+0.0309r^3z-0.0287rz^2;$$

$$w=0.0456r^2-0.0077r^4-0.4148z+0.0287r^2z+0.0257z^2+0.0399z^3;$$

$$R=-2.0145r-1.6178rz;\ Z=-0.25r^2-1.4893z.$$

Similarly, other particular solutions to the problem can be obtained using different types of polynomials, for example, $z^\alpha y^\beta+z^\alpha$, etc. to form the basis.

The approach under study provides for lots of solutions of one boundary value problem of the elasticity theory in the presence of mass forces. Displacements and mass forces whose combinations give the distribution of stresses satisfying the given forces at the boundary are also subject. The first method is devoid of this feature because mass forces in it are part of the given conditions, and the task is only to find the field of displacements.

Discussion and Conclusions. The second method to solve the problem has the best convergence. In addition, unlike the first method, when calculating scalar products in the process of orthogonalization and in determining the Fourier coefficients, the second method does not use deformations and displacements. Here, a basic set of stresses is generated, and its trace at the boundary is a basis set of surface forces with which orthogonalization and series construction are performed. This means that when calculating scalar products, there is no error associated with the components responsible for the rigid displacement that may occur during the formation of the basis [11].

If problems with mass forces are considered, then the second method may be useful in the formation of many particular solutions whose stresses satisfy certain conditions at the boundary. These solutions can be used as the basis ones for a more complex problem, and also be useful under determining the elastic fields realized from fictitious loads resulting from the application of the Poincare method [7, 8].

The accuracy of solving problems of the elasticity theory by the boundary-state method is analyzed using different approaches to the construction of scalar products. The solution to the problem of the linear elasticity theory using

the representation of a general solution to the Lame equation in the form of a Fourier series on basis functions and expression (6) as the orthogonalizer of these functions had the best convergence.

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About the author

Ivanychev, Dmitrii A., associate professor of the General Mechanics Department, Lipetsk State Technical University (30, st. Moskovskaya, Lipetsk, 398055, RF), Cand.Sci. (Phys.-Math.), ORCID: http://orcid.org/0000-0002-7736-9311, Lsivdmal @mail.ru

The author has read and approved the final manuscript.